Moduli Spaces

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One of the most central, fundamental and classic themes in Algebraic Geometry is the classification problem. Given a collection of geometric objects, and an equivalence relation among them, we would like to find parameters to determine the equivalence classes. A few of these parameters may be numerical: dimension and genus of an algebraic variety, degree and rank of vector bundles. However, most of them form a larger, uncountable space, often an algebraic variety: Plücker coordinates of subspaces of a vector space, j-invariants of elliptic curves, and so on.

The parameters are what Riemann termed "moduli," in connection to smooth projective curves, a word that stuck with algebraic geometers and is now used in far more generality. The functorial point of view inaugurated in Algebraic Geometry by Grothendieck allowed us to define precisely what moduli spaces are. In addition, the introduction of methods from topology, mainly sheaves and their cohomology, first by Serre and then, in the needed generality, by Grothendieck, and a new and more geometric approach by Mumford to an old subject, Invariant Theory, have made it possible to construct many moduli spaces, opening the way for a thriving theory.

The new approach and methods developed are often considered very hard to be grasped by ordinary doctor students. The main reference for them, Hartshorne's book listed below, does not often do a good job in motivating the new techniques. The applications therein given to curves and surfaces could, for the most part, have been carried out using a more classical theory.

A course is thus proposed to introduce the fundamental methods for constructing moduli spaces in algebraic geometry, and at the same time introduce Grothendieck's techniques to students. The layout of the course, broadly speaking, is this: First, I review the classical algebraic methods of the 19th century German school for dealing with invariants, and the revolution created by Hilbert with his theorem on the finite generation of them. Second, I talk about Mumford's more geometric approach, of expanding on Hilbert's methods to construct quotients of varieties by group actions. Third, I cover the basic theory of the Grassmann varieties, first by the classical approach and then by the functorial point of view. Fourth, sheaves and Cech cohomology are introduced, and basic theorems about them are proved. Fifth, I show how one can construct the Hilbert scheme, parameterizing subschemes of a given variety, as a subvariety of an appropriate Grassmann variety, using the methods just developed. The so-called Quot scheme is also presented. Sixth, I show how to use Hilbert schemes and the Invariant Theory developed by Mumford to construct the moduli space of smooth curves envisioned by Riemann. The compactification of this moduli space by stable curves is also presented. Finally, I present Jacobians and their compactifications, my main topic of research, reviewing the work done so far by various people, and introducing the recent approach by Amini and myself to produce a compactification of the relative Jacobian over the moduli of stable curves.

The course is structured in 30 lectures of 50 minutes each. They are divided by topics in the following way:

1. The classification problem in algebraic geometry
2. Invariant theory of homogeneous forms
3. The geometry of group actions
4. The Hilbert–Nagata theorem on finite generation of invariants
5. Geometric Invariant Theory (GIT): stability and semistability
6. GIT: The Hilbert–Mumford criterion
7. GIT: Homogeneous forms revisited
8. Grassmannians
9. Functors of points
10. Vector bundles

11. The Grassmannian revisited
12. Sheaves
13. (Cech) Cohomology
14. Cohomology of the projective space
15. Continuity in algebraic geometry: flatness
16. Semicontinuity and base change
17. The Castelnuovo–Mumford regularity
18. Hilbert schemes
19. Moduli of smooth curves I
20. Moduli of smooth curves II
21. (Generalized) Jacobians
22. Compactifications of Jacobians I
23. Compactifications of Jacobians II

The main prerequisite for the course is knowledge of basic algebraic geometry, the basic theory of varieties, as taught in Chapter 1 of Hartshorne's book or in Shafarevich's. For the second half of the course, some knowledge of Commutative and Homological Algebra, and of the theory of schemes, sheaves and their cohomology, as done in Chapters 2 and 3 of Hartshorne's book, is useful, but not strictly necessary as I will review what is necessary. References for the course are:

1. E. Esteves, *Construção de espaços de moduli.* IMPA, 1997.
2. E. Esteves, *Moduli spaces.* Notes, in progress.
3. J. Fogarty and D. Mumford, *Geometric Invariant Theory.* Springer, Berlin, 1982.
4. R. Hartshorne, *Algebraic Geometry.* Springer, New York, 1977.
5. G. Kempf, *Algebraic Varieties.* London Math. Soc. Lecture Notes, Cambridge, 1993.
6. D. Mumford, *Lectures on curves on an algebraic surface.* Princeton, 1966.
7. P. Newstead, *Introduction to moduli problems and orbit spaces.* Tata Lecture Notes, Springer, 1978.
8. J. L. Potier, *Fibrés vectoriels sur les courbes algébriques.* Paris, 1995.
9. E. Sernesi, *Topics on families of projective schemes.* Queen's, Kingston, 1986.
10. I. Shafarevich, *Basic Algebraic Geometry.* Springer, Berlin, 1994.

Notes for each lecture will be made available on a weekly basis. They cover the topics given in each lecture, and contain proofs of almost all the statements made.